be in either the liquid or solid phase, is at a uniform temperature T_s. It is further assumed that the presence of condensed oxides on the particle surface in no way inhibits the successful consumption of the chemical reactants, implying these condensed oxides either exist in liquid phase such that they contract under surface tension and leave large part of surface exposed, or are sufficiently porous to the diffusion of the oxidizer gas if they are in the solid phase. Finally we assume that the thermal conductivity coefficients λ are constants, that the heat capacities c for the gas and condensed phases are constants and equal to each other, and that the Lewis number is unity.

The diffusion equation for the oxidizer gas can be written as

$$\nu_2 M Y_O - \hat{r}^2 dY_O/d\hat{r} = -\nu_1 M \quad (1)$$

where

$$\nu_2 = \sigma(1+\nu_1) - \nu_1 \quad (2)$$

and σ is equal to 1 or 0 when the oxides are formed in the gas phase and transported outward or in the condensed phase and reside on the surface.

Integrating Eq. (1) with the boundary condition that at $\hat{r}=1$, $Y_O=0$, we obtain

$$Y_O = (\nu_1/\nu_2)\{\exp[-\nu_2 M(\hat{r}^{-1}-1)]-1\} \quad (3)$$

Evaluating Eq. (3) at $Y_O(\infty) = Y_{O,\infty}$, the constant, nondimensionalized mass consumption rate can be found as

$$M = \nu_2^{-1}\ln[1+(\nu_2/\nu_1)Y_{O,\infty}] \quad (4)$$

Putting Eq. (4) into Eq. (3), the radial distribution of the oxidizer gas can be expressed explicitly as

$$Y_O = (\nu_1/\nu_2)\{[1+(\nu_2/\nu_1)Y_{O,\infty}]^{-(\hat{r}^{-1}-1)}-1\} \quad (5)$$

The gas-phase energy equation can be written as

$$\nu_2 M(\hat{T}-\hat{T}_s) - \hat{r}^2 d\hat{T}/d\hat{r} = M\hat{Q} \quad (6)$$

where

$$\hat{Q} = \hat{Q}_1 + (1-\sigma)\hat{Q}_2 - 1 \quad (7)$$

Integrating Eq. (6) with the boundary condition that at $\hat{r}=1$, $\hat{T}=\hat{T}_s$, and using Eq. (4), we obtain

$$\hat{T} = \hat{T}_s + (\hat{Q}/\nu_2)\{1-[1+(\nu_2/\nu_1)Y_{O,\infty}]^{-(\hat{r}^{-1}-1)}\} \quad (8)$$

The surface temperature $\hat{T}_s$ is found by evaluating Eq. (8) at $\hat{T}(\infty) = \hat{T}_\infty$

$$\hat{T}_s = \hat{T}_\infty + \hat{Q}Y_{O,\infty}/\nu_1 \quad (9)$$

Substituting Eq. (9) into Eq. (8), the radial distribution of temperature can be explicitly expressed as

$$\hat{T} = \hat{T}_\infty + (\hat{Q}/\nu_2)\{1+(\nu_2/\nu_1)Y_{O,\infty}\}\{1-[1+(\nu_2/\nu_1)Y_{O,\infty}]^{-1/\hat{r}}\} \quad (10)$$

Equations (4, 5, 9 and 10) provide the complete solutions to the problem.

Discussions

The present model is expected to be applicable to an atmosphere satisfying the relation $\hat{T}_s < \hat{T}_b$, or

$$\hat{T}_b > \hat{T}_\infty + \hat{Q}Y_{O,\infty}/\nu_1 \quad (11)$$

where T_b is the boiling point of the metal. When the surface temperature reaches the boiling point, vaporization rate is greatly enhanced, therefore favoring vapor-phase combustion. It may be noted that the upper limit of this surface reaction model, as given by Eq. (11) with $T_b = T_s$, is just the flammability limit identified for the vapor-phase combustion model of Refs. 1 and 2.

The value σ assumes is to be determined in a consistent manner. If, for example, by assuming $\sigma = 1$ a value for T_s is obtained which is lower than the oxides' boiling point, then $\sigma = 0$ should be used. It is expected, however, that a unique value of σ exists for a given pair of reactants.

Finally we note that since the rate of supply of the oxidizer is specified independent of T_∞, and since the oxidizer reacts stoichiometrically with the fuel, the fuel burning rate M is found to be independent of T_∞.

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Inclusion of Transverse Shear Deformation in Finite Element Displacement Formulations

R. Narayanaswami* and H. M. Adelman†

NASA Langley Research Center, Hampton, Va.

INCLUDING transverse shear deformation in finite-element formulations is essential if the elements are used to model such components as thick plates, short beams, or sandwich panels consisting of soft cores and stiff face sheets. In a recent publication, Irons and Razzaque[1] presented an artificial, and in their own words "theoretically inadequate," method to include transverse shear in a plate-bending finite element as an afterthought. The stated justification for their somewhat complicated procedure is the idea put forth by Severn[2] that the usual finite-element technique of directly minimizing the strain energy is not applicable when transverse shear deformation is included.

It is the purpose of the present Note to demonstrate that a straightforward energy minimization does in fact yield the correct finite-element behavior when transverse shear effects are included and that additional finite-element grid point degrees of freedom are not required to treat these effects. Since the derivation in Ref. 1 have their basis in Ref. 2, many of the comments herein are directed at Ref. 2.

The stiffness matrix for a beam element with transverse shear deformation will be derived in this section in a similar fashion to Ref. 2 to illustrate the correct approach for a simple and familiar component. The general approach to plate elements is described in Refs. 3 and 4.

The normal displacement w in the element is the same as that used in Ref. 2, thus

$$w = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad (1)$$

The relation between transverse shear strain γ, w', and θ is, from Fig. 1

$$w' = \theta + \gamma \quad (2)$$

Transverse shear strain γ, consistent with the cubic polynomial for w, is assumed to be independent of x, i.e.,

$$\gamma = b_0 \quad (3)$$

Moment equilibrium of an infinitesimal beam section yields

$$dM/dx - V = 0 \quad (4)$$

Note that the introduction of equilibrium equations in the formulation is formally equivalent to a constraint, since a dis-

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* NASA-NRC Resident Research Associate.

† Research Scientist, Thermal Structures Branch, Structures and Dynamics Division.

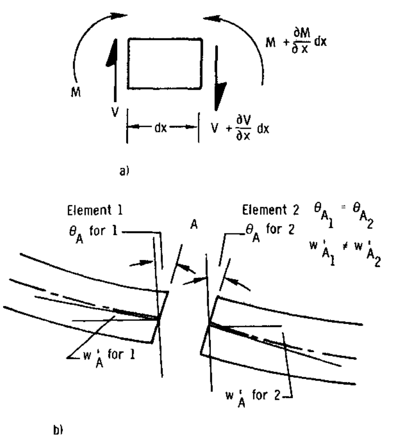

Fig. 1 Beam finite-element notations; a) beam element; b) two beam elements meeting at point A.

placement approach does not normally rely on the explicit satisfaction of equilibrium equations. This step, however, facilitates inclusion of transverse shear in displacement formulations without involving additional grid point degrees of freedom.

The bending moment-curvature relation is

$$M = -EI\, d\theta/dx \tag{5}$$

The shearing force V is related to the transverse shear strain γ by

$$V = kGA\gamma \tag{6}$$

where k is the shear factor.

Using Eqs. (1–3, 5, 6) in Eq. (4), gives

$$b_0 = -6a_3 g \tag{7}$$

where $g = EI/kGA$.

Using Eqs. (1–3, and 7) allows θ to be expressed as a polynomial in x as follows:

$$\theta = a_1 + 2a_2 x + (3x^2 - 6g)a_3 \tag{8}$$

Using Eqs. (1) and (8), it is now possible to express the coefficients a_0 through a_3 in terms of the displacement w and the rotation θ of the beam at the ends $x = 0$ and $x = a$. Letting the subscript 1 refer to the end $x = 0$ and the subscript 2 refer to $x = a$, the result is

$$\begin{aligned} a_0 &= w_1 \\ a_1 &= \frac{1}{a^2+12g}\left[-\frac{12g}{a^2}w_1 + (a^2+6g)\theta_1 + \frac{12g}{a}w_2 - 6g\theta_2\right] \\ a_2 &= \frac{1}{a^2+12g}\left[-3w_1 - \frac{(2a^2+6g)}{a}\theta_1 + 3w_2 - \frac{(a^2-6g)}{a}\theta_2\right] \\ a_3 &= \frac{1}{a^2+12g}\left(\frac{2}{a}w_1 + \theta_1 - \frac{2}{a}w_2 + \theta_2\right) \end{aligned} \tag{9}$$

Substituting these into Eq. (1) yields

$$\begin{aligned} w = {} & \left[1 - \frac{12g}{a^2(a^2+12g)}x - \frac{3}{a^2+12g}x^2 + \frac{2}{a(a^2+12g)}x^3\right]w_1 + \\ & \left[\frac{a^2+6g}{a^2+12g}x - \frac{2a^2+6g}{a(a^2+12g)}x^2 + \frac{1}{a^2+12g}x^3\right]\theta_1 + \\ & \left[\frac{12g}{a(a^2+12g)}x + \frac{3}{a^2+12g}x^2 - \frac{2}{a(a^2+12g)}x^3\right]w_2 + \\ & \left[\frac{-6g}{a^2+12g}x + \frac{6g-a^2}{a(a^2+12g)}x^2 + \frac{1}{a^2+12g}x^3\right]\theta_2 \end{aligned} \tag{10}$$

The expression for strain energy is

$$U = \frac{EI}{2}\int_0^a \left(\frac{d\theta}{dx}\right)^2 dx + \frac{kGA}{2}\int_0^a \gamma^2\, dx \tag{11}$$

Substituting Eqs. (3 and 7–9) into Eq. (11) gives the correct beam stiffness matrix in terms of w_1, θ_1, w_2, and θ_2

$$[k] = \frac{12}{a^2+12g}\frac{EI}{a}\begin{bmatrix} 1 & & & \text{sym} \\ \dfrac{a}{2} & \dfrac{a^2}{3}+g & & \\ -1 & \dfrac{-a}{2} & 1 & \\ \dfrac{a}{2} & \dfrac{a^2}{6}-g & \dfrac{-a}{2} & \dfrac{a^2}{3}+g \end{bmatrix} \tag{12}$$

It is important to note that retention of θ as the rotational degree of freedom, rather than w' as in Refs. 1 and 2, is crucial. (Reference 2 uses the symbol θ to refer to w'.) This is because in order to combine two or more elements as in Fig. 1b, it is necessary to enforce continuity of θ and not w' (w' is continuous only in the absence of transverse shear deformation). In addition, enforcement of a built-in edge condition requires that $\theta = 0$ (not $w' = 0$).

The equation in Ref. 2 corresponding to Eq. (10) is

$$\begin{aligned} w = {} & \left(\frac{2x^3}{a^3} - \frac{3x^2}{a^2} + 1\right)w_1 + \left(\frac{x^3}{a^2} - \frac{2x^2}{a} + x\right)w_1' + \\ & \left(-\frac{2x^3}{a^3} + \frac{3x^2}{a^2}\right)w_2 + \left(\frac{x^3}{a^2} - \frac{x^2}{a}\right)w_2' \end{aligned} \tag{13}$$

Equation (13) is a valid interpolation formula if it is recognized that w' is given by Eq. (2). However, in Ref. 2 the tacit assumption that $w' = \theta$ has led to an erroneous stiffness matrix.

References 1 and 2 are not the only places where assumption that $w' = \theta$ is used. For example, in a more recent attempt to account for transverse shear effects in a beam element, Nickell and Secor[5] developed two Timoshenko beam elements TIM7 and TIM4. TIM7 needs 3 additional degrees of freedom to control transverse shear whereas TIM4 is derived using a method similar to that used in this paper and suggested by Egle.[6] However, the TIM4 element retains w' rather than θ as the rotational degree of freedom. In fact, Eq. (25) of Ref. 5 is the same as the stiffness matrix [Eq. (9) of Ref. 2]. Nickell and Secor used the TIM4 element to solve a cantilever beam vibration problem which yielded poor results. Using the stiffness matrix in Eq. (11) based on the correct choice of degrees of freedom (w_1, θ_1, w_2, θ_2), the authors have observed that the convergence characteristics for the frequencies of the same beam are vastly improved and competitive with that of TIM7.

Conclusions

In conclusion, the following points are emphasized.

1) A straightforward energy minimization approach will yield the correct stiffness matrix in displacement formulations when transverse shear effects are included (without additional degrees of freedom to treat transverse shear).

2) In any finite element displacement formulation where transverse shear deformations are to be included, it is essential that the rotation of the normal (and not the derivative of w) be retained as a grid point degree of freedom. The inability of the TIM4 beam element[5] to represent the geometric boundary conditions for a cantilever beam illustrates this assertion.

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